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PIEZOELECTRIC SOLID WITH AN ELLIPTIC INCLUSION OR HOLE

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Abstract—The two-dimensional problem of an elliptic hole in a solid of anisotropic piezoelectric material is studied. The Stroh formalism is adopted here. Real form solutions are obtained along the hole boundary in the case of an arbitrarily prescribed vector field on the hole surface. For an elliptic rigid inclusion of electric conductor subjected to a line force, a torque, and a line charge, a real form solution at the interface is obtained. Finally, general solutions for an elliptic piezoelectric inclusion with uniform loading at infinity are investigated. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

In 1958 and 1962, Stroh elaborated the work of Eshelby *et al.* (1953) on two-dimensional problems of general anisotropic elasticity involving dislocations, line forces, and steady waves. This powerful and elegant approach was named the Stroh formalism.

In 1975, Barnett and Lothe extended Stroh's 1962 paper to include the piezoelectric effect in which an eight-dimensional framework had been developed. Here, we consider the two-dimensional problem of an elliptic hole in a solid of anisotropic piezoelectric material. Similar problems had been studied by Pak (1992) and Sosa (1991). Although some useful solutions had been derived in these two papers, they were both restricted to the transversely isotropic situation. In Pak's 1992 paper, special remote loading conditions were employed and the concentration effect was studied. Likewise, only remote loadings were considered in Sosa's 1991 paper.

Here, solutions of an arbitrarily prescribed loading on the hole surface are derived. Furthermore, in the case of an elliptic rigid inclusion of electric conductor subjected to a line force, a torque, and a free line charge, real form solutions along the elliptic interface are obtained which could be used to examine the concentration effect. Finally, we investigate the situation of an elliptic piezoelectric inclusion with uniform loading at infinity.

In the following basic solutions of the Stroh formalism with the piezoelectric effect are given. Some boundary conditions are shown in Section 2. In Sections 3 and 4, a few useful relations are derived. General field solutions to the elliptic problem are obtained in Section 5 with emphasis on solutions along the elliptic boundary. Such boundary solutions could be employed to investigate the concentration effect. However, arbitrary constant vectors are involved and remain unknown. They will be determined in Sections 6, 7, and 8 in which different boundary conditions are applied.

In a Cartesian coordinate system (x_1, x_2, x_3) the constitutive equations for piezoelectric materials are given by (Tiersten, 1969)

$$\sigma_{ij} = C_{ijkm}u_{k,m} + e_{mij}\varphi_{,m}, \quad D_i = e_{ikm}u_{k,m} - \omega_{im}\varphi_{,m} \quad (i,j,k,m = 1,2,3) \tag{1}$$

in which repeated indices mean summation and a comma stands for partial differentiation. σ_{ij} is the elastic stress and D_i is the electric displacement. Coefficients C_{ijkm}, e_{mij}, ω_{im} are, respectively, the elastic stiffnesses, piezoelectric constants, and permittivities with the following symmetries:

$$C_{ijkm} = C_{jikm} = C_{kmij}, \quad e_{mij} = e_{mji}, \quad \omega_{im} = \omega_{mi}. \tag{2}$$

u_k is the elastic displacement and φ is the electrostatic potential. C_{ijkm} and ω_{im} are positive definite in the sense that

$$C_{ijkm}u_{i,j}u_{k,m} > 0, \quad \omega_{im}E_iE_m > 0 \tag{3}$$

for arbitrary real nonzero $u_{i,j}$ and E_i with

$$E_i = -\varphi_{,i}. \tag{4}$$

In the absence of body forces and free charges, the balance laws require

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0. \tag{5}$$

For two-dimensional deformations in which u_k and φ depend on x_1 and x_2 only, a general solution to (5) is given by

$$u_J = a_J f(z) \quad (J = 1,2,3,4) \tag{6}$$

in which

$$z = x_1 + px_2, \quad u_4 = \varphi, \tag{7}$$

and p, a_J are constants to be determined. In matrix notation,

$$\mathbf{u} = \mathbf{a}f(z). \tag{8}$$

Thus $\mathbf{u}$, $\mathbf{a}$ are four-vectors and $\mathbf{u}$ is called the generalized displacement. By defining

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^E & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\omega_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^E & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\omega_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\omega_{22} \end{bmatrix}, \tag{9}$$

where

$$(\mathbf{Q}^E)_{ik} = C_{i1k1}, \quad (\mathbf{R}^E)_{ik} = C_{i1k2}, \quad (\mathbf{T}^E)_{ik} = C_{i2k2}, \quad (\mathbf{e}_{ij})_m = e_{ijm}, \tag{10}$$

we combine (1), (5), and (6) into one equation as

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0}. \tag{11}$$

The 4×4 matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric but not positive definite. However, they can be shown to be nonsingular.

Let the generalized stress function vector $\boldsymbol{\phi}$ be defined as

$$\boldsymbol{\phi} = \mathbf{b}f(z), \quad \mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = \frac{-1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{12}$$

with

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \quad D_1 = -\phi_{4,2}, \quad D_2 = \phi_{4,1}. \tag{13}$$

The second equality in $(12)_2$ follows from (11). Equation (13) provides all components of σ_{ij} and D_i except σ_{33} and D_3; they can be determined from (1).

With the positive definiteness of C_{ijkm} and ω_{im} shown in (3), the eigenvalues p of (11) are all complex and consist of four pairs of complex conjugates. Let

$$p_{\alpha+4} = \bar{p}_\alpha, \quad \operatorname{Im}\{p_\alpha\} > 0 \quad (\alpha = 1, 2, 3, 4), \tag{14}$$

$$\mathbf{a}_{\alpha+4} = \bar{\mathbf{a}}_\alpha, \quad \mathbf{b}_{\alpha+4} = \bar{\mathbf{b}}_\alpha, \tag{15}$$

where the overbars denote the complex conjugates. The general solution obtained by superposing eight solutions of (8) and $(12)_1$ associated with the eight eigenvalues p_α are

$$\mathbf{u} = 2\operatorname{Re}\left\{\sum_{\alpha=1}^{4} \mathbf{a}_\alpha f_\alpha(z_\alpha)\right\}, \quad \boldsymbol{\phi} = 2\operatorname{Re}\left\{\sum_{\alpha=1}^{4} \mathbf{b}_\alpha f_\alpha(z_\alpha)\right\} \tag{16}$$

in which Re stands for the real part and $f_{\alpha+4} = \bar{f}_\alpha$ ($\alpha = 1, 2, 3, 4$) is chosen.

In most applications

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \quad (\alpha \text{ not summed}) \tag{17}$$

is assumed. Hence, eqn (16) reduces to, in matrix notation,

$$\mathbf{u} = 2\operatorname{Re}\{\mathbf{A}\langle f(z_*)\rangle\mathbf{q}\}, \quad \boldsymbol{\phi} = 2\operatorname{Re}\{\mathbf{B}\langle f(z_*)\rangle\mathbf{q}\} \tag{18}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 4×4 matrices given by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4], \tag{19}$$

and $\langle f(z_*)\rangle$ is the 4×4 diagonal matrix

$$\langle f(z_*)\rangle = \operatorname{diag}\langle f(z_1), f(z_2), f(z_3), f(z_4)\rangle. \tag{20}$$

The elements of the four-vector $\mathbf{q}$ are q_α ($\alpha = 1, 2, 3, 4$). Notice that the solutions given in (18) are in terms of the arbitrary function $f(z_\alpha)$ and the arbitrary complex constant vector $\mathbf{q}$.

2. BOUNDARY CONDITIONS

Consider an arc or a contour C described by

$$C(s): \begin{cases} x_1 = x_1(s) \\ x_2 = x_2(s) \end{cases}, \quad [x_1'(s)]^2 + [x_2'(s)]^2 = 1, \tag{21}$$

where s is the arc-length. The unit tangential vector $\mathbf{n}$ and the unit normal vector $\mathbf{m}$ are given by

$$\mathbf{n}^{\mathrm{T}} = \left[\frac{\mathrm{d}x_1}{\mathrm{d}s}, \frac{\mathrm{d}x_2}{\mathrm{d}s}, 0\right], \quad \mathbf{m}^{\mathrm{T}} = \left[-\frac{\mathrm{d}x_2}{\mathrm{d}s}, \frac{\mathrm{d}x_1}{\mathrm{d}s}, 0\right], \tag{22}$$

respectively. By taking derivative of $\boldsymbol{\phi}$ in the direction of increasing s (with material on the RIGHT-hand side) and using (13), we obtain

$$\frac{\mathrm{d}\phi_j}{\mathrm{d}s} = t_j \ (j = 1, 2, 3), \quad \frac{\mathrm{d}\phi_4}{\mathrm{d}s} = \mathbf{D}\cdot\mathbf{m} = D_m, \tag{23}$$

in which t_j is the component of surface traction vector. Similarly, one obtains

$$\frac{\mathrm{d}u_4}{\mathrm{d}s} = -\mathbf{E}\cdot\mathbf{n} = -E_n. \tag{24}$$

If we consider a dielectric interface with materials indicated by "1" and "2", the electrical conditions at the interface are

$$\mathbf{E}_1\cdot\mathbf{n} = \mathbf{E}_2\cdot\mathbf{n}, \quad \mathbf{D}_1\cdot\mathbf{m}_1 + \mathbf{D}_2\cdot\mathbf{m}_2 = \sigma_s \tag{25}$$

where $\mathbf{n}$ is a unit vector tangential to the dielectric interface, $\mathbf{m}_i$ is an inward normal unit vector, and σ_s is the free surface charge density along the interface. Without loss in generality, we can rewrite (25) as

$$\varphi_1 = \varphi_2, \quad \mathbf{D}_1\cdot\mathbf{m}_1 + \mathbf{D}_2\cdot\mathbf{m}_2 = \sigma_s. \tag{26}$$

If we have an interface between electric conductor "1" and dielectric "2", then inside the electric conductor,

$$\mathbf{D}_1 = \mathbf{0}, \quad \mathbf{E}_1 = \mathbf{0}. \tag{27}$$

In the dielectric, at the interface

$$\mathbf{E}_2\cdot\mathbf{n} = E_{2n} = 0, \quad \mathbf{D}_2\cdot\mathbf{m}_2 = \sigma_s. \tag{28}$$

3. EIGHT-DIMENSIONAL FORMALISM

The two equations in $(12)_2$ can be rewritten as

$$\begin{bmatrix} -\mathbf{R}^T & \mathbf{I} \\ -\mathbf{Q} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = p\begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}. \tag{29}$$

Since $\mathbf{T}^{-1}$ exists, we can reduce (29) to

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{30}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{31}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \tag{32}$$

The real 4×4 matrices $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric. Equation (30) is a standard eigenrelation in the eight-dimensional space. There are eight eigenvalues p_α ($\alpha = 1, 2, \ldots, 8$) and eight associated eigenvectors $\boldsymbol{\xi}_\alpha$. The eigenvalues are the roots of the determinant

$$\|\mathbf{N} - p\mathbf{I}\| = 0. \tag{33}$$

The vector $\boldsymbol{\xi}$ in (30) is a right eigenvector. The left eigenvector $\boldsymbol{\eta}$ is defined by

$$\boldsymbol{\eta}^T\mathbf{N} = p\boldsymbol{\eta}^T, \quad \mathbf{N}^T\boldsymbol{\eta} = p\boldsymbol{\eta}, \tag{34}$$

and can be shown to be

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}. \tag{35}$$

Normalization of $\boldsymbol{\xi}_\alpha$ and $\boldsymbol{\eta}_\beta$ (which are orthogonal to each other) gives

$$\boldsymbol{\eta}_\beta^{\mathrm{T}} \boldsymbol{\xi}_\alpha = \delta_{\beta\alpha} \tag{36}$$

where $\delta_{\beta\alpha}$ is the Kronecker delta. Making use of (15), (19), $(31)_2$, and (35), eqn (36) is written as

$$\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{37}$$

This is the orthogonality relation. The two 8×8 matrices on the left hand side of (37) are the inverses of each other. Their product commutes so that

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{38}$$

This is the closure relation and is equivalent to

$$\mathbf{A}\mathbf{B}^{\mathrm{T}} + \bar{\mathbf{A}}\bar{\mathbf{B}}^{\mathrm{T}} = \mathbf{I} = \mathbf{B}\mathbf{A}^{\mathrm{T}} + \bar{\mathbf{B}}\bar{\mathbf{A}}^{\mathrm{T}}, \quad \mathbf{A}\mathbf{A}^{\mathrm{T}} + \bar{\mathbf{A}}\bar{\mathbf{A}}^{\mathrm{T}} = \mathbf{0} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \bar{\mathbf{B}}\bar{\mathbf{B}}^{\mathrm{T}}. \tag{39}$$

Hence, the three matrices **S**, **H**, **L** defined by

$$\mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = i2\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -i2\mathbf{B}\mathbf{B}^{\mathrm{T}} \tag{40}$$

are real. The matrices **H** and **L** are symmetric and nonsingular (Lothe and Barnett, 1976).

Since **S**, **H**, **L** are real, the following relation exists (Chung, 1995; Ting and Yan, 1991)

$$\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{41}$$

which indicates that **SH** and **LS** are anti-symmetric. It can be shown that $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{S}\mathbf{L}^{-1}$ are also anti-symmetric.

Finally, we rewrite (30) as

$$\mathbf{N} \begin{bmatrix} \mathbf{a} f'(z) \\ \mathbf{b} f'(z) \end{bmatrix} = \begin{bmatrix} \mathbf{a} p f'(z) \\ \mathbf{b} p f'(z) \end{bmatrix}. \tag{42}$$

Employing $(7)_1$, (8), and $(12)_1$ leads to a matrix differential equation for **u** and $\boldsymbol{\phi}$,

$$\mathbf{N} \begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{,2} \\ \boldsymbol{\phi}_{,2} \end{bmatrix}. \tag{43}$$

4. THE INTEGRAL FORMALISM

Let the tensor E_{iJKm} be defined by (Barnett and Lothe, 1975; Kuo and Barnett, 1991)

$$\begin{aligned} E_{iJKm} &= C_{ijkm} \quad (J, K = 1, 2, 3), \\ &= e_{mij} \quad (J = 1, 2, 3; K = 4), \\ &= e_{ikm} \quad (J = 4; K = 1, 2, 3), \\ &= -\omega_{im} \quad (J = K = 4). \end{aligned} \tag{44}$$

With $\mathbf{n}(\omega)$ and $\mathbf{m}(\omega)$ given by

$$\mathbf{n}^{\mathrm{T}}(\omega) = [\cos\omega, \ \sin\omega, \ 0], \quad \mathbf{m}^{\mathrm{T}}(\omega) = [-\sin\omega, \ \cos\omega, \ 0] \tag{45}$$

in which ω is a real parameter range from 0 to 2π, we let

$$\begin{aligned} Q_{JK}(\omega) &= n_i(\omega)E_{iJKm}n_m(\omega), \quad R_{JK}(\omega) = n_i(\omega)E_{iJKm}m_m(\omega), \\ T_{JK}(\omega) &= m_i(\omega)E_{iJKm}m_m(\omega), \end{aligned} \tag{46}$$

and

$$\begin{aligned} \mathbf{N}_1(\omega) &= -\mathbf{T}^{-1}(\omega)\mathbf{R}^{\mathrm{T}}(\omega), \quad \mathbf{N}_2(\omega) = \mathbf{T}^{-1}(\omega), \\ \mathbf{N}_3(\omega) &= \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^{\mathrm{T}}(\omega) - \mathbf{Q}(\omega). \end{aligned} \tag{47}$$

Lothe and Barnett (1976) have shown that

$$\mathbf{S} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}_1(\omega)\,\mathrm{d}\omega, \quad \mathbf{H} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}_2(\omega)\,\mathrm{d}\omega, \quad -\mathbf{L} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}_3(\omega)\,\mathrm{d}\omega. \tag{48}$$

Equations (48) provide an alternate to (40) for the Barnett-Lothe tensors **S**, **H**, and **L**.

5. THE GENERAL SOLUTIONS

In this section the general solutions for two-dimensional deformations along an elliptic boundary will be derived (Ting and Yan, 1991). An ellipse Γ given by

$$\Gamma: \begin{cases} x_1 = a\cos\psi \\ x_2 = b\sin\psi \end{cases} \tag{49}$$

is shown in Fig. 1. Let **n** and **m** be the unit vectors tangential and normal to the elliptic boundary, respectively and ω be the angle between vector **n** and the positive x_1 axis. Hence,

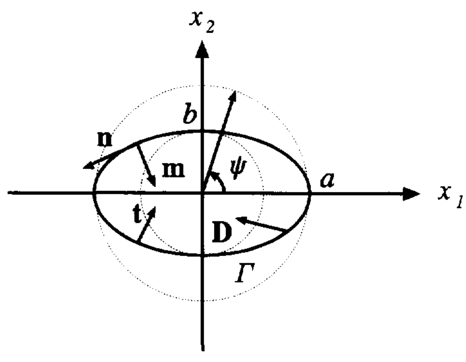

Fig. 1. An ellipse in the (x_1, x_2) plane.

$$\mathbf{n}^{\mathrm{T}} = [\cos\omega, \sin\omega, 0], \quad \mathbf{m}^{\mathrm{T}} = [-\sin\omega, \cos\omega, 0] \tag{50}$$

which is (45). The infinitesimal arc-length ds of the ellipse is given by

$$\mathrm{d}s = \rho(\psi)\,\mathrm{d}\psi, \quad \rho(\psi) = \sqrt{a^2\sin^2\psi + b^2\cos^2\psi}. \tag{51}$$

From $(22)_1$ and (49) we see that

$$\cos\omega = \frac{-a}{\rho(\psi)}\sin\psi, \quad \sin\omega = \frac{b}{\rho(\psi)}\cos\psi \tag{52}$$

when comparison is made with $(50)_1$.

If there is a line force $\mathbf{f}$ and a free line-charge density λ applied at the origin (Fig. 1), by employing (23), the equilibrium conditions give

$$\left.\begin{aligned} &\oint_\Gamma (\mathbf{t}_m)_j\,\mathrm{d}s = \lim_{B\to A} \phi_j(B) - \phi_j(A) = f_j \quad (j = 1,2,3) \\ &\oint_\Gamma \mathbf{D}\cdot\mathbf{m}\,\mathrm{d}s = \lim_{B\to A} \phi_4(B) - \phi_4(A) = -\lambda \end{aligned}\right\} \tag{53}$$

in which $\mathbf{t}_m$ and $\mathbf{D}$ are the surface traction and the electric displacement of the medium along the elliptic boundary Γ, respectively. Therefore, we have a jump in $\boldsymbol{\phi}$ across the positive x_1 axis if $\mathbf{f}$ and λ are not equal to zero. Points A and B are in fact the same point on positive x_1 axis except that when one moves from A to B counter-clockwise, the whole ellipse Γ is transversed.

Consider the transformation

$$z_\alpha = c_\alpha\zeta_\alpha + d_\alpha\zeta_\alpha^{-1} \quad (\alpha = 1,2,3,4), \tag{54}$$

where c_α and d_α are complex constants and $z_\alpha = x_1 + p_\alpha x_2$. The constants c_α and d_α are chosen such that when $(x_1, x_2) \subset \Gamma$, ζ_α ($\alpha = 1, 2, 3, 4$) is on a unit circle. That is,

$$\zeta_\alpha|_\Gamma = \mathrm{e}^{i\psi} = \cos\psi + i\sin\psi \quad (\alpha = 1,2,3,4) \tag{55}$$

when $z_\alpha = a\cos\psi + p_\alpha b\sin\psi$ and one obtains

$$c_\alpha = \frac{a - ip_\alpha b}{2}, \quad d_\alpha = \frac{a + ip_\alpha b}{2}. \tag{56}$$

Since a, b, and Im $\{p_\alpha\}$ are all positive and non-zero, it can be shown that the branch points ζ_α of the transformation (54) are located inside the unit circle in the ζ_α-plane. Hence, the branch points in the (x_1, x_2) plane are located inside the ellipse. In addition, the transformation is one-to-one outside the elliptic hole.

In order to satisfy the jump conditions stated in (53) along the positive x_1 axis, the arbitrary function $f(z_\alpha)$ given in (18) is chosen to be

$$f(z_\alpha) = \ln\zeta_\alpha, \tag{57}$$

with z_α and ζ_α being related by (54). Also, by putting (Ting, 1986; 1988a, Hwu and Ting, 1989)

$$\mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{g}_0 + \mathbf{B}^{\mathrm{T}}\mathbf{h}_0 \tag{58}$$

in (18) where $\mathbf{g}_0$ and $\mathbf{h}_0$ are real constants, we obtain the first basic solution

$$\left.\begin{aligned}\mathbf{u}^{\mathrm{I}} &= 2\,\mathrm{Re}\,\{\mathbf{A}\langle\ln\zeta_*\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_0 + 2\,\mathrm{Re}\,\{\mathbf{A}\langle\ln\zeta_*\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_0\\ \boldsymbol{\phi}^{\mathrm{I}} &= 2\,\mathrm{Re}\,\{\mathbf{B}\langle\ln\zeta_*\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_0 + 2\,\mathrm{Re}\,\{\mathbf{B}\langle\ln\zeta_*\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_0\end{aligned}\right\} \tag{59}$$

in which $\langle\ln\zeta_*\rangle$ is the diagonal matrix of $\ln\zeta_\alpha$ with $\alpha = 1, 2, 3, 4$. Since $\ln\zeta_\alpha$ is a multi-valued function, a cut along $\psi = 0$ is introduced which makes $\mathbf{u}^{\mathrm{I}}$, $\boldsymbol{\phi}^{\mathrm{I}}$ single-valued and allows a discontinuity along the positive x_1 axis. As $z_\alpha \to \infty$, the elastic stresses and the electric displacements obtained from $(59)_2$ vanish. This is consistent with the boundary conditions at infinity.

In order to provide analytical solutions outside the ellipse,

$$f(z_\alpha) = \zeta_\alpha^{-k}, \quad \mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{g}_k + \mathbf{B}^{\mathrm{T}}\mathbf{h}_k, \quad (k = 1, 2, \ldots) \tag{60}$$

are assumed in (18) where $\mathbf{g}_k$, $\mathbf{h}_k$ are real constants. Superimposing the solutions from $k = 1$ to ∞ leads to the second basic solution

$$\left.\begin{aligned}\mathbf{u}^{\mathrm{II}} &= 2\sum_{k=1}^{\infty}\mathrm{Re}\,\{\mathbf{A}\langle\zeta_*^{-k}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_k + 2\sum_{k=1}^{\infty}\mathrm{Re}\,\{\mathbf{A}\langle\zeta_*^{-k}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_k\\ \boldsymbol{\phi}^{\mathrm{II}} &= 2\sum_{k=1}^{\infty}\mathrm{Re}\,\{\mathbf{B}\langle\zeta_*^{-k}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_k + 2\sum_{k=1}^{\infty}\mathrm{Re}\,\{\mathbf{B}\langle\zeta_*^{-k}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_k\end{aligned}\right\} \tag{61}$$

in which $\langle\zeta_*^{-k}\rangle$ is the diagonal matrix of ζ_α^{-k} $(\alpha = 1, 2, 3, 4)$. Notice that both $\mathbf{u}^{\mathrm{II}}$ and $\boldsymbol{\phi}^{\mathrm{II}}$ approach zero as $z_\alpha \to \infty$ (or $\zeta_\alpha \to \infty$).

With (55) it is easy to see that

$$\langle\ln\zeta_*|_\Gamma\rangle = i\psi\mathbf{I}, \quad \langle\zeta_*^{-k}|_\Gamma\rangle = \cos(k\psi)\mathbf{I} - i\sin(k\psi)\mathbf{I}. \tag{62}$$

Substituting back in (59) gives the first basic solution along the elliptic boundary Γ as

$$\mathbf{u}^{\mathrm{I}}|_\Gamma = \psi\hat{\mathbf{h}}_0, \quad \boldsymbol{\phi}^{\mathrm{I}}|_\Gamma = \psi\hat{\mathbf{g}}_0, \tag{63}$$

$$\hat{\mathbf{h}}_0 = \mathbf{H}\mathbf{g}_0 + \mathbf{S}\mathbf{h}_0, \quad \hat{\mathbf{g}}_0 = \mathbf{S}^{\mathrm{T}}\mathbf{g}_0 - \mathbf{L}\mathbf{h}_0, \tag{64}$$

when using (40). Similarly, the second basic solution along the elliptic boundary Γ is in the form

$$\mathbf{u}^{\mathrm{II}}|_\Gamma = \sum_{k=1}^{\infty}[\cos(k\psi)\mathbf{h}_k - \sin(k\psi)\hat{\mathbf{h}}_k], \quad \boldsymbol{\phi}^{\mathrm{II}}|_\Gamma = \sum_{k=1}^{\infty}[\cos(k\psi)\mathbf{g}_k - \sin(k\psi)\hat{\mathbf{g}}_k] \tag{65}$$

in which

$$\hat{\mathbf{h}}_k = \mathbf{H}\mathbf{g}_k + \mathbf{S}\mathbf{h}_k, \quad \hat{\mathbf{g}}_k = \mathbf{S}^{\mathrm{T}}\mathbf{g}_k - \mathbf{L}\mathbf{h}_k. \tag{66}$$

Some useful relations between $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, and $\hat{\mathbf{h}}_k$ are given below (Chung, 1995; Ting and Yan, 1991).

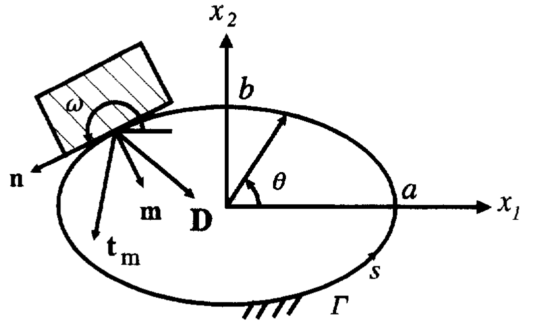

Fig. 2. Generalized stress vector along the elliptic boundary.

$$\mathbf{h}_k = \mathbf{L}^{-1}(\mathbf{S}^{\mathrm{T}}\mathbf{g}_k - \hat{\mathbf{g}}_k), \quad \hat{\mathbf{h}}_k = \mathbf{L}^{-1}(\mathbf{g}_k + \mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_k) \quad (k = 0, 1, 2, \ldots), \tag{67}$$

$$\hat{\mathbf{g}}_k = -\mathbf{H}^{-1}(\mathbf{h}_k + \mathbf{S}\hat{\mathbf{h}}_k), \quad \mathbf{g}_k = -\mathbf{H}^{-1}(\mathbf{S}\mathbf{h}_k - \hat{\mathbf{h}}_k) \quad (k = 0, 1, 2, \ldots). \tag{68}$$

In fact, any two of $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, and $\hat{\mathbf{h}}_k$ can be written in terms of the others.

In order to investigate the concentration effect, we will derive the generalized stress vector $\hat{\mathbf{t}}_m$ and the generalized hoop stress vector $\hat{\mathbf{t}}_n$ along the elliptic boundary. In Fig. 2, if n is the arc-length of Γ measured in the direction of $\mathbf{n}$, then from (23) the generalized stress vector $\hat{\mathbf{t}}_m$ is defined as

$$\hat{\mathbf{t}}_m^{\mathrm{T}} = [(\mathbf{t}_m)_1, (\mathbf{t}_m)_2, (\mathbf{t}_m)_3, D_m] = \boldsymbol{\phi}_{,n}^{\mathrm{T}}, \tag{69}$$

or, using $(51)_1$,

$$\hat{\mathbf{t}}_m = \boldsymbol{\phi}_{,n} = \frac{\partial \boldsymbol{\phi}|_\Gamma}{\rho(\psi)\,\partial\psi}. \tag{70}$$

Substituting $\boldsymbol{\phi}^{\mathrm{I}}|_\Gamma$ and $\boldsymbol{\phi}^{\mathrm{II}}|_\Gamma$ given in $(63)_2$ and $(65)_2$ leads to

$$\hat{\mathbf{t}}_m^{\mathrm{I}} = \frac{1}{\rho(\psi)}\hat{\mathbf{g}}_0, \quad \hat{\mathbf{t}}_m^{\mathrm{II}} = \frac{-1}{\rho(\psi)}\sum_{k=1}^{\infty} k[\sin(k\psi)\mathbf{g}_k + \cos(k\psi)\hat{\mathbf{g}}_k]. \tag{71}$$

Note that the arbitrary constant vectors $\hat{\mathbf{g}}_0$, $\mathbf{g}_k$, and $\hat{\mathbf{g}}_k$ $(k = 1, 2, \ldots)$ are involved.

Similarly, in Fig. 3, if the generalized hoop stress vector $\hat{\mathbf{t}}_n$ is defined by

$$\hat{\mathbf{t}}_n^{\mathrm{T}} = [(\mathbf{t}_n)_1, (\mathbf{t}_n)_2, (\mathbf{t}_n)_3, -D_n] \tag{72}$$

with $-D_n = \mathbf{D}\cdot(-\mathbf{n})$, then by letting m be the arc-length measured in the direction of $\mathbf{m}$, it is clear that

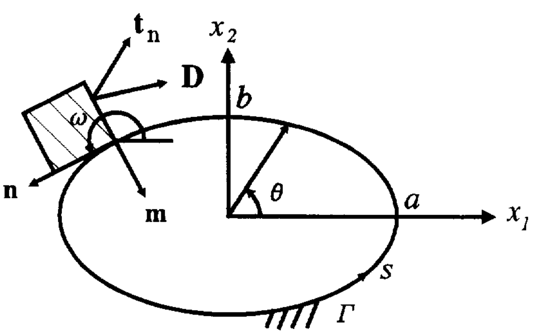

Fig. 3. Generalized hoop stress vector along the elliptic boundary.

$$\hat{\mathbf{t}}_n = \boldsymbol{\phi}_{,m} = -\boldsymbol{\phi}_{,1}\sin\omega + \boldsymbol{\phi}_{,2}\cos\omega \tag{73}$$

where use has been made of (23) and $(50)_2$. Alternatively, one can express $\hat{\mathbf{t}}_n$ in terms of $\mathbf{u}_{,n}$ and $\hat{\mathbf{t}}_m$ as (Chung, 1995; Ting and Yan, 1991)

$$\hat{\mathbf{t}}_n = \mathbf{N}_3(\omega)\mathbf{u}_{,n} + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_m \tag{74}$$

which is a relation that applies to a general shape of boundary.

For the elliptic boundary Γ shown in Fig. 3, we have

$$\hat{\mathbf{t}}_n = \mathbf{N}_3(\omega)\frac{\partial \mathbf{u}|_\Gamma}{\rho(\psi)\,\partial\psi} + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_m = \hat{\mathbf{t}}_n^{\mathrm{I}} + \hat{\mathbf{t}}_n^{\mathrm{II}} \tag{75}$$

in which

$$\left.\begin{aligned} \hat{\mathbf{t}}_n^{\mathrm{I}} &= \frac{1}{\rho(\psi)}[\mathbf{N}_3(\omega)\hat{\mathbf{h}}_0 + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{g}}_0] \\ \hat{\mathbf{t}}_n^{\mathrm{II}} &= -\frac{\mathbf{N}_3(\omega)}{\rho(\psi)}\sum_{k=1}^{\infty}\{k[\sin(k\psi)\mathbf{h}_k + \cos(k\psi)\hat{\mathbf{h}}_k]\} \\ &\quad -\frac{\mathbf{N}_1^{\mathrm{T}}(\omega)}{\rho(\psi)}\sum_{k=1}^{\infty}\{k[\sin(k\psi)\mathbf{g}_k + \cos(k\psi)\hat{\mathbf{g}}_k]\} \end{aligned}\right\}, \tag{76}$$

with the use of $(63)_1$, $(65)_1$, and (71). Again the arbitrary constant vectors are involved.

In the case of a hole with free surface and electrically open (i.e., zero normal component of electric displacement), (74) then takes the form

$$\hat{\mathbf{t}}_n = \mathbf{N}_3(\omega)\mathbf{u}_{,n} \tag{77}$$

and $\hat{\mathbf{t}}_m = \mathbf{0}$ (Kuo and Barnett, 1991).

If we have a rigid inclusion of electric conductor with boundary condition $\mathbf{E}\cdot\mathbf{n} = 0$ given by $(28)_1$, it follows from (24) that (74) is reduced to

$$\hat{\mathbf{t}}_n = \mathbf{N}_3(\omega)\begin{bmatrix}\mathbf{u}_{r,n} \\ 0\end{bmatrix} + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_m \tag{78}$$

in which $\mathbf{u}_r$ is the rigid body motion of the boundary. Hence,

$$\mathbf{u}_r = \mathbf{u}_0 + \Omega\mathbf{e}_3 \times \mathbf{r}_\Gamma \tag{79}$$

where $\mathbf{u}_0$ is a rigid body translation, Ω is the rotation about x_3 axis and $\mathbf{r}_\Gamma$ is the position vector of a point on the boundary. For an elliptic boundary Γ, $\mathbf{r}_\Gamma = a\cos\psi\mathbf{e}_1 + b\sin\psi\mathbf{e}_2$. Thus,

$$\mathbf{r}_{\Gamma,n} = \mathbf{n}, \quad \mathbf{u}_{r,n} = \Omega\mathbf{e}_3 \times \mathbf{r}_{\Gamma,n} = \Omega\mathbf{m}. \tag{80}$$

Substituting $(80)_2$ into (78) yields (Chung, 1995)

$$\hat{\mathbf{t}}_n = \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_m \tag{81}$$

which is also applicable to circular boundary.

The hoop stress σ_{nn} is given by

$$\sigma_{nn} = \mathbf{t}_n \cdot (-\mathbf{n}) \tag{82}$$

and the two shear stresses are

$$\sigma_{nm} = \mathbf{t}_n \cdot (-\mathbf{m}), \quad \sigma_{n3} = \mathbf{t}_n \cdot (-\mathbf{e}_3) \tag{83}$$

in which $\mathbf{e}_3^{\mathrm{T}} = [0, 0, 1]$.

Notice that our solutions are all in terms of arbitrary constant vectors $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, $\hat{\mathbf{h}}_k$ with $k = 0, 1, 2, \ldots$ When we determine these constants, we have the solutions. In the next three sections, the arbitrary constant vectors will be determined by applying appropriate boundary conditions.

6. AN ELLIPTIC HOLE

We consider an elliptic hole shown in Fig. 2. Let

$$\left.\begin{aligned} \mathbf{u}|_\Gamma &= \mathbf{u}^{\mathrm{I}}|_\Gamma + \mathbf{u}^{\mathrm{II}}|_\Gamma = \psi\hat{\mathbf{h}}_0 + \sum_{k=1}^{\infty} [\cos(k\psi)\mathbf{h}_k - \sin(k\psi)\hat{\mathbf{h}}_k] \\ \boldsymbol{\phi}|_\Gamma &= \boldsymbol{\phi}^{\mathrm{I}}|_\Gamma + \boldsymbol{\phi}^{\mathrm{II}}|_\Gamma = \psi\hat{\mathbf{g}}_0 + \sum_{k=1}^{\infty} [\cos(k\psi)\mathbf{g}_k - \sin(k\psi)\hat{\mathbf{g}}_k] \end{aligned}\right\}. \tag{84}$$

The right hand sides of (84) are given by (63) and (65). Since $\mathbf{u}|_\Gamma$ must be single-valued, it follows from $(84)_1$ that

$$\hat{\mathbf{h}}_0 = \mathbf{0}, \tag{85}$$

and one obtains

$$\mathbf{h}_0 = -\mathbf{S}^{-1}\mathbf{H}\mathbf{g}_0 \tag{86}$$

when $(68)_2$ is employed. The generalized stress vector along the elliptic hole boundary Γ is, with (71),

$$\hat{\mathbf{t}}_m = \hat{\mathbf{t}}_m^{\mathrm{I}} + \hat{\mathbf{t}}_m^{\mathrm{II}} = \frac{1}{\rho(\psi)}\hat{\mathbf{g}}_0 - \frac{1}{\rho(\psi)}\sum_{k=1}^{\infty} \{k[\sin(k\psi)\mathbf{g}_k + \cos(k\psi)\hat{\mathbf{g}}_k]\}. \tag{87}$$

To find $\hat{\mathbf{t}}_n$, we first substitute (67) into $(84)_1$. With (85) and

$$\mathbf{S}\mathbf{L}^{-1} + \mathbf{L}^{-1}\mathbf{S}^{\mathrm{T}} = \mathbf{0}, \tag{88}$$

the generalized displacement vector along Γ becomes

$$\mathbf{u}|_\Gamma = -\mathbf{S}\mathbf{L}^{-1}\sum_{k=1}^{\infty} [\cos(k\psi)\mathbf{g}_k - \sin(k\psi)\hat{\mathbf{g}}_k] - \mathbf{L}^{-1}\sum_{k=1}^{\infty} [\cos(k\psi)\hat{\mathbf{g}}_k + \sin(k\psi)\mathbf{g}_k]. \tag{89}$$

Equation (88) comes from the anti-symmetric property of $\mathbf{S}\mathbf{L}^{-1}$. From $(75)_1$ the generalized hoop stress vector then takes the form

$$\hat{\mathbf{t}}_n = [\mathbf{N}_1^{\mathrm{T}}(\omega) - \mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}]\hat{\mathbf{t}}_m + \frac{\mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}}{\rho(\psi)}\hat{\mathbf{g}}_0 + \frac{\mathbf{N}_3(\omega)\mathbf{L}^{-1}}{\rho(\psi)}\sum_{k=1}^{\infty}\{k[\sin(k\psi)\hat{\mathbf{g}}_k - \cos(k\psi)\mathbf{g}_k]\} \quad (90)$$

in which $(87)_2$ and (89) are employed. Alternatively, one obtains

$$\hat{\mathbf{t}}_n = \mathbf{G}_1(\omega)\hat{\mathbf{t}}_m + \frac{\mathbf{G}_3(\omega)}{\rho(\psi)}\left\{\mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_0 - \sum_{k=1}^{\infty}\{k[\sin(k\psi)\hat{\mathbf{g}}_k - \cos(k\psi)\mathbf{g}_k]\}\right\} \quad (91)$$

with

$$\mathbf{G}_1(\omega) = \mathbf{N}_1^{\mathrm{T}}(\omega) - \mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}, \quad \mathbf{G}_3(\omega) = -\mathbf{N}_3(\omega)\mathbf{L}^{-1}. \quad (92)$$

Since $\mathbf{N}_3(\omega)$ is symmetric so is $\mathbf{G}_3(\omega)\mathbf{L}$. It can be shown that

$$\mathbf{G}_1(\omega)\mathbf{L} = \mathbf{N}_1^{\mathrm{T}}(\omega)\mathbf{L} - \mathbf{N}_3(\omega)\mathbf{S} \quad (93)$$

is also symmetric.

For an arbitrarily prescribed boundary condition along Γ, we define a four-vector $\hat{\boldsymbol{\tau}}_m(\psi)$,

$$\hat{\boldsymbol{\tau}}_m^{\mathrm{T}}(\psi) = [(\boldsymbol{\tau}_m)_1(\psi),\ (\boldsymbol{\tau}_m)_2(\psi),\ (\boldsymbol{\tau}_m)_3(\psi),\ \tilde{D}_m(\psi)] = [\boldsymbol{\tau}_m^{\mathrm{T}}(\psi),\ \tilde{D}_m(\psi)]$$

in which $(\boldsymbol{\tau}_m)_i(\psi)$ $(i = 1, 2, 3)$ are arbitrarily prescribed traction components on Γ while the stress at infinity vanishes. $\tilde{D}_m(\psi)$ $(= \tilde{\mathbf{D}}(\psi)\cdot\mathbf{m})$ is an arbitrarily prescribed normal component of electric displacement of the medium along Γ with the electric displacement vanishing at infinity also. Notice that $\tilde{D}_m(\psi) = 0$ refers to the so-called electrically opened situation (Kuo and Barnett, 1991; Pak, 1992). With the arbitrarily prescribed boundary conditions it is clear that

$$\hat{\boldsymbol{\tau}}_m(\psi) = \hat{\mathbf{t}}_m. \quad (94)$$

Employing $(87)_2$ and the orthogonality properties between sine and cosine, some of the arbitrary constant vectors in terms of $\hat{\boldsymbol{\tau}}_m(\psi)$ are determined as

$$\left.\begin{aligned} \hat{\mathbf{g}}_0 &= \frac{1}{2\pi}\int_0^{2\pi}\rho(\psi)\hat{\boldsymbol{\tau}}_m(\psi)\,\mathrm{d}\psi \\ \mathbf{g}_k &= \frac{-1}{k\pi}\int_0^{2\pi}\rho(\psi)\hat{\boldsymbol{\tau}}_m(\psi)\sin(k\psi)\,\mathrm{d}\psi \quad (k \geqslant 1) \\ \hat{\mathbf{g}}_k &= \frac{-1}{k\pi}\int_0^{2\pi}\rho(\psi)\hat{\boldsymbol{\tau}}_m(\psi)\cos(k\psi)\,\mathrm{d}\psi \quad (k \geqslant 1) \end{aligned}\right\}. \quad (95)$$

Equations $(67)_2$, (85), and $(95)_1$ can then be used to find $\mathbf{g}_0$. After that, $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ are obtained by employing $(67)_1$ and $(67)_2$, respectively. In fact $\mathbf{h}_0$ can simply be computed from (86).

Indeed, $\hat{\mathbf{g}}_0$ is related to the resultant line force $\mathbf{f}$ applied on Γ and the resultant free line-charge density λ enclosed by Γ. By considering (53) and (94), the equilibrium equation becomes

$$\int_0^{2\pi} \hat{\boldsymbol{\tau}}_m(\psi)\rho(\psi)\,\mathrm{d}\psi = \oint_\Gamma \hat{\boldsymbol{\tau}}_m(\psi)\,\mathrm{d}n = 2\pi\hat{\mathbf{g}}_0 = \begin{bmatrix} \mathbf{f} \\ -\lambda \end{bmatrix} = \hat{\mathbf{f}} \tag{96}$$

where n is the arc-length along the elliptic hole boundary Γ.

In the following some special boundary conditions of $\hat{\boldsymbol{\tau}}_m(\psi)$ are considered. We first assume

$$\rho(\psi)\tilde{D}_m(\psi) = \beta = \text{constant}. \tag{97}$$

By (96), one has

$$\tilde{D}_m(\psi) = \frac{\beta}{\rho(\psi)} = \frac{-\lambda}{2\pi\rho(\psi)}. \tag{98}$$

Suppose that a uniform pressure p is applied along the elliptic boundary Γ. i.e.,

$$\boldsymbol{\tau}_m(\psi) = -p\mathbf{m}(\omega). \tag{99}$$

With this and $(98)_1$, the prescribed generalized stress vector takes the form

$$\hat{\boldsymbol{\tau}}_m^{\mathrm{T}}(\psi) = \left[p\frac{b}{\rho(\psi)}\cos\psi, p\frac{a}{\rho(\psi)}\sin\psi, 0, \frac{\beta}{\rho(\psi)}\right] \tag{100}$$

in which $(50)_2$ and (52) are employed. By comparing with $(87)_2$ the arbitrary constant vectors $\mathbf{g}_k$ and $\hat{\mathbf{g}}_k$ are easily determined as

$$\hat{\mathbf{g}}_0 = \beta\tilde{\mathbf{e}}_4, \quad \mathbf{g}_1 = -pa\tilde{\mathbf{e}}_2, \quad \hat{\mathbf{g}}_1 = -pb\tilde{\mathbf{e}}_1, \quad \mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0} \quad (k \geqslant 2), \tag{101}$$

where the four-vectors $\tilde{\mathbf{e}}_I$ are defined as

$$(\tilde{\mathbf{e}}_I)_J = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases} \quad (I, J = 1,2,3,4). \tag{102}$$

Thus, with (89), the generalized displacement vector along the elliptic boundary is simply

$$\mathbf{u}|_\Gamma = \mathbf{S}\mathbf{L}^{-1}p(x_1\tilde{\mathbf{e}}_2 - x_2\tilde{\mathbf{e}}_1) + \mathbf{L}^{-1}p\left(\frac{b}{a}x_1\tilde{\mathbf{e}}_1 + \frac{a}{b}x_2\tilde{\mathbf{e}}_2\right). \tag{103}$$

Similarly, with (91), (94), and (100), the generalized hoop stress vector is given by

$$\hat{\mathbf{t}}_n = \mathbf{G}_1(\omega)\begin{bmatrix} p\sin\omega \\ -p\cos\omega \\ 0 \\ \dfrac{\beta}{\rho(\psi)} \end{bmatrix} + \mathbf{G}_3(\omega)\mathbf{S}^{\mathrm{T}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\beta}{\rho(\psi)} \end{bmatrix} - \mathbf{G}_3(\omega)p\begin{bmatrix} \dfrac{b}{a}\cos\omega \\ \dfrac{a}{b}\sin\omega \\ 0 \\ 0 \end{bmatrix}. \tag{104}$$

Likewise, we can consider a uniform in-plane shear stress τ instead of a uniform pressure p.

In addition to the special boundary condition stated in (97), the traction boundary condition described by

$$\rho(\psi)\tau_m(\psi) = \gamma = \text{constant} = \frac{\mathbf{f}}{2\pi} \tag{105}$$

is considered. Following the similar procedure given above, we have

$$\mathbf{u}|_\Gamma = \mathbf{0} = \text{constant} \tag{106}$$

which implies that the elliptic hole is *not* distorted and the electrostatic potential is *constant* on the hole surface. Consequently, if the elliptic hole is filled with a rigid electric conductor and subjected to a concentrated line force **f** and a free line charge density λ at the origin, the generalized stress vector $\hat{\mathbf{t}}_m$ along the interface is simply given by (97) and (105) (Ting and Yan, 1991). The generalized hoop stress vector is

$$\hat{\mathbf{t}}_n = \frac{1}{\rho(\psi)}[\mathbf{G}_1(\omega)+\mathbf{G}_3(\omega)\mathbf{S}^{\mathrm{T}}]\begin{bmatrix}\gamma\\ \beta\end{bmatrix} \tag{107}$$

which can be shown to be consistent with what is stated in (81) when (88) and (92) are employed.

In the following the solutions for boundary conditions prescribed as

$$\hat{\mathbf{t}}_m(\psi) = \begin{bmatrix}\tilde{\sigma}_{11}m_1+\tilde{\sigma}_{21}m_2+\tilde{\sigma}_{31}m_3\\ \tilde{\sigma}_{12}m_1+\tilde{\sigma}_{22}m_2+\tilde{\sigma}_{32}m_3\\ \tilde{\sigma}_{13}m_1+\tilde{\sigma}_{23}m_2+\tilde{\sigma}_{33}m_3\\ \tilde{D}_1m_1+\tilde{D}_2m_2+\tilde{D}_3m_3\end{bmatrix} = \begin{bmatrix}\tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\\ \tilde{\mathbf{D}}^{\mathrm{T}}\end{bmatrix}\mathbf{m}(\omega) = \begin{bmatrix}\tilde{\boldsymbol{\sigma}}\\ \tilde{\mathbf{D}}^{\mathrm{T}}\end{bmatrix}\mathbf{m}(\omega) \tag{108}$$

will be derived. Here, $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\mathbf{D}}$ are the prescribed *uniform* stress field and electric displacement field along the elliptic hole boundary Γ within the medium, respectively. Since $\mathbf{m}^{\mathrm{T}}(\omega) = [-\sin\omega, \cos\omega, 0]$, we have from (52)

$$\hat{\mathbf{t}}_m(\psi) = \frac{-b}{\rho(\psi)}\cos\psi\tilde{\mathbf{t}}_1 - \frac{a}{\rho(\psi)}\sin\psi\tilde{\mathbf{t}}_2,\quad \tilde{\mathbf{t}}_1 = \begin{bmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{21}\\ \tilde{\sigma}_{31}\\ \tilde{D}_1\end{bmatrix},\quad \tilde{\mathbf{t}}_2 = \begin{bmatrix}\tilde{\sigma}_{12}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{32}\\ \tilde{D}_2\end{bmatrix}. \tag{109}$$

The arbitrary constants are determined by comparing with $(87)_2$ which leads to

$$\hat{\mathbf{g}}_0 = \mathbf{0},\quad \mathbf{g}_1 = a\tilde{\mathbf{t}}_2,\quad \hat{\mathbf{g}}_1 = b\tilde{\mathbf{t}}_1,\quad \mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0}\quad (k \geqslant 2). \tag{110}$$

With the use of (49) and (110), (89) reduces to

$$\mathbf{u}|_\Gamma = -\mathbf{S}\mathbf{L}^{-1}[x_1\tilde{\mathbf{t}}_2 - x_2\tilde{\mathbf{t}}_1] - \mathbf{L}^{-1}\left[\frac{b}{a}x_1\tilde{\mathbf{t}}_1 + \frac{a}{b}x_2\tilde{\mathbf{t}}_2\right]. \tag{111}$$

The generalized hoop stress vector stated in (91) takes the form

$$\hat{\mathbf{t}}_n = \mathbf{G}_1(\omega)[\cos\omega\tilde{\mathbf{t}}_2 - \sin\omega\tilde{\mathbf{t}}_1] + \mathbf{G}_3(\omega)\left[\frac{b}{a}\cos\omega\tilde{\mathbf{t}}_1 + \frac{a}{b}\sin\omega\tilde{\mathbf{t}}_2\right] \tag{112}$$

when (52), (94), $(109)_1$ and (110) are employed.

For the problem of an elliptic hole subject to a uniform stress field $\boldsymbol{\sigma}^\infty$ and a uniform electric displacement field $\mathbf{D}^\infty$ at infinity while the surface of the hole is free of traction and electrically open (Pak, 1992), the solution may be separated into two parts. The first part is the uniform solution in which the stress and electric displacement are $\boldsymbol{\sigma}^{\infty}$ and $\mathbf{D}^{\infty}$ everywhere. The second part is the "disturbed" state due to the presence of the hole. The solution of the second part must satisfy the boundary conditions that the stress and electric displacement vanish at infinity while at the hole surface

$$\hat{\mathbf{t}}_m(\psi) = \begin{bmatrix} -\boldsymbol{\sigma}^{\infty} \\ -\mathbf{D}^{\infty} \end{bmatrix} \mathbf{m}(\omega). \tag{113}$$

This is precisely the problem considered in this section.

In general, for an arbitrarily prescribed boundary condition $\hat{\mathbf{t}}_m(\psi)$ the series solutions $\mathbf{u}|_{\Gamma}$ and $\hat{\mathbf{t}}_n$ given above retain infinite terms. However, by introducing the conjugate function (Bary, 1964; Ting and Yan, 1991), one can rewrite the infinite series solutions in terms of definite integrals.

7. A RIGID INCLUSION OF ELECTRIC CONDUCTOR

In Section 6 the solutions for a rigid elliptic inclusion of electric conductor in the absence of torque are studied. Here, in addition to a line force $\mathbf{f}$ and a free line-charge density λ, a counter-clockwise torque $T\mathbf{e}_3$ is applied. The generalized stress function vector and generalized displacement vector along the elliptic boundary Γ are, respectively,

$$\boldsymbol{\phi}|_{\Gamma} = \psi\hat{\mathbf{g}}_0 + \sum_{k=1}^{\infty} [\cos(k\psi)\mathbf{g}_k - \sin(k\psi)\hat{\mathbf{g}}_k], \quad \mathbf{u}|_{\Gamma} = \sum_{k=1}^{\infty} [\cos(k\psi)\mathbf{h}_k - \sin(k\psi)\hat{\mathbf{h}}_k]. \tag{114}$$

The equilibrium conditions stated in (53) are,

$$-\oint_{\Gamma} \hat{\mathbf{t}}_m \, \mathrm{d}n + \hat{\mathbf{f}} = \mathbf{0}, \quad \hat{\mathbf{f}} = \begin{bmatrix} \mathbf{f} \\ -\lambda \end{bmatrix} \tag{115}$$

and $\hat{\mathbf{t}}_m$ is defined in $(69)_1$. Using $(70)_2$ and $(114)_1$, (115) reduces to

$$-\int_0^{2\pi} \frac{\partial\boldsymbol{\phi}|_{\Gamma}}{\rho(\psi)\,\partial\psi} \rho(\psi)\,\mathrm{d}\psi + \hat{\mathbf{f}} = \boldsymbol{\phi}(0)|_{\Gamma} - \boldsymbol{\phi}(2\pi)|_{\Gamma} + \hat{\mathbf{f}} = -2\pi\hat{\mathbf{g}}_0 + \hat{\mathbf{f}} = \mathbf{0}, \tag{116}$$

which is the same result as given in (96) so that $\hat{\mathbf{g}}_0$ can be computed. Since the rigid inclusion does not deform and the electrostatic potential φ $(= u_4)$ is constant along the elliptic boundary Γ, by ignoring the constant components and noticing that $\mathbf{r}_{\Gamma} = a\cos\psi\mathbf{e}_1 + b\sin\psi\mathbf{e}_2$, we have, by using (79),

$$\mathbf{u}|_{\Gamma} = \Omega(a\cos\psi\tilde{\mathbf{e}}_2 - b\sin\psi\tilde{\mathbf{e}}_1). \tag{117}$$

Some of the arbitrary constants are determined by comparing with $(114)_2$ which yields

$$\mathbf{h}_1 = \Omega a\tilde{\mathbf{e}}_2, \quad \hat{\mathbf{h}}_1 = \Omega b\tilde{\mathbf{e}}_1, \quad \mathbf{h}_k = \hat{\mathbf{h}}_k = 0 \quad (k \geqslant 2). \tag{118}$$

The constants $\hat{\mathbf{g}}_k$ and $\mathbf{g}_k$ are then obtained from $(68)_1$ and $(68)_2$, respectively. Note that the angular rotation Ω is still unknown. To determine it, we consider the equilibrium of moment which gives (Chung, 1995; Ting and Yan, 1991)

$$\Omega = \frac{T}{\pi U}, \tag{119}$$

where

$$U = b^2 H_{11}^{-1} + a^2 H_{22}^{-1} + 2ab(H^{-1}S)_{21},$$
$$= \bar{\mathbf{c}}^{\mathrm{T}}(\mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S})\mathbf{c} = \bar{\mathbf{c}}^{\mathrm{T}}\mathbf{Z}\mathbf{c}. \tag{120}$$

In the above $\mathbf{c}^{\mathrm{T}} = [-ib, a, 0, 0]$ and $\mathbf{Z}$ is the surface impedance tensor (Lothe and Barnett, 1976; Kuo and Barnett, 1991). Hence, one obtains

$$\hat{\mathbf{t}}_m = \frac{1}{\rho(\psi)}\frac{\hat{\mathbf{f}}}{2\pi} + \frac{T}{\pi U}\mathbf{H}^{-1}\left[\frac{a}{b}\sin\omega\tilde{\mathbf{e}}_2 + \frac{b}{a}\cos\omega\tilde{\mathbf{e}}_1 - \mathbf{S}(\cos\omega\tilde{\mathbf{e}}_2 - \sin\omega\tilde{\mathbf{e}}_1)\right]. \tag{121}$$

For a circular rigid inclusion $\rho(\psi) = a = b = l$, (121) is simplified to

$$\hat{\mathbf{t}}_m = \frac{1}{2\pi l}\hat{\mathbf{f}} + \frac{T}{\pi U}\mathbf{H}^{-1}[\cos\omega\tilde{\mathbf{e}}_1 + \sin\omega\tilde{\mathbf{e}}_2 + \mathbf{S}(\sin\omega\tilde{\mathbf{e}}_1 - \cos\omega\tilde{\mathbf{e}}_2)]. \tag{122}$$

In the case of zero torque, one obtains from (121)

$$\hat{\mathbf{t}}_m = \frac{1}{\rho(\psi)}\frac{\hat{\mathbf{f}}}{2\pi}. \tag{123}$$

This means that $\rho(\psi)\hat{\mathbf{t}}_m$ is a constant which is consistent with our observation in (97) and (105). Furthermore, for a circular rigid inclusion, (122) reduces to

$$\hat{\mathbf{t}}_m = \begin{bmatrix} \mathbf{t}_m \\ D_m \end{bmatrix} = \frac{1}{2\pi l}\begin{bmatrix} \mathbf{f} \\ -\lambda \end{bmatrix}. \tag{124}$$

This means that the traction $\mathbf{t}_m$ along the circular interface is a constant in the direction of $\mathbf{f}$. This phenomena was also observed in the purely elastic case (Ting and Yan, 1991).

To find the generalized hoop stress vector $\hat{\mathbf{t}}_n$, (81) is employed in which $\hat{\mathbf{t}}_m$ is given in (121) and (122) for elliptic and circular rigid inclusion of electric conductor, respectively.

8. A PIEZOELECTRIC INCLUSION

In this section we consider an elliptic piezoelectric inclusion within a piezoelectric matrix. If the matrix is subjected to uniform fields at infinity, with either $\varepsilon_{ij}^{\infty}, E_i^{\infty}$ or $\sigma_{ij}^{\infty}, D_i^{\infty}$ prescribed such that $\varepsilon_{33}^{\infty} = 0$ and $E_3^{\infty} = 0$, then the uniform field solutions in the absence of an elliptic inclusion can be written as (Hwu and Ting, 1989)

$$\mathbf{u}^{\infty} = x_1\boldsymbol{\varepsilon}_1^{\infty} + x_2\boldsymbol{\varepsilon}_2^{\infty}, \quad \boldsymbol{\phi}^{\infty} = x_1\mathbf{t}_2^{\infty} - x_2\mathbf{t}_1^{\infty}, \tag{125}$$

in which

$$\boldsymbol{\varepsilon}_1^{\infty} = \mathbf{u}_{,1}^{\infty} = \begin{bmatrix} \varepsilon_{11}^{\infty} \\ 0 \\ 2\varepsilon_{13}^{\infty} \\ -E_1^{\infty} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^{\infty} = \mathbf{u}_{,2}^{\infty} = \begin{bmatrix} 2\varepsilon_{21}^{\infty} \\ \varepsilon_{22}^{\infty} \\ 2\varepsilon_{23}^{\infty} \\ -E_2^{\infty} \end{bmatrix} \tag{126}$$

and

$$\mathbf{t}_1^{\infty} = [\sigma_{11}^{\infty}, \sigma_{12}^{\infty}, \sigma_{13}^{\infty}, D_1^{\infty}]^{\mathrm{T}} = -\boldsymbol{\phi}_{,2}^{\infty}, \quad \mathbf{t}_2^{\infty} = [\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty}]^{\mathrm{T}} = \boldsymbol{\phi}_{,1}^{\infty}. \tag{127}$$

Note that $(125)_1$ is unique up to a rigid body motion and a constant electrostatic potential. If $\varepsilon_{ij}^{\infty}, E_i^{\infty}$ are given, $\sigma_{ij}^{\infty}, D_i^{\infty}$ can be obtained by using (1), (4), and

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{128}$$

If $\sigma_{ij}^{\infty}, D_i^{\infty}$ are known, the following constitutive equations are employed (Sosa, 1991):

$$\varepsilon_{ij} = S_{ijks}^{D}\sigma_{ks}+g_{kij}D_k, \quad E_i = -g_{iks}\sigma_{ks}+\beta_{ik}^{\sigma}D_k. \tag{129}$$

The zero element in $\boldsymbol{\varepsilon}_1^{\infty}$ implies that there is no rotation of the x_1 axis with respect to the x_3 axis. By means of superposition the field solutions in the piezoelectric matrix are (Hwu and Ting, 1989)

$$\left.\begin{aligned}\mathbf{u} &= \mathbf{u}^{\infty}+2\,\mathrm{Re}\,\{\mathbf{A}\langle\zeta_*^{-1}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_1+2\,\mathrm{Re}\,\{\mathbf{A}\langle\zeta_*^{-1}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_1\\ \boldsymbol{\phi} &= \boldsymbol{\phi}^{\infty}+2\,\mathrm{Re}\,\{\mathbf{B}\langle\zeta_*^{-1}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_1+2\,\mathrm{Re}\,\{\mathbf{B}\langle\zeta_*^{-1}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_1\end{aligned}\right\}. \tag{130}$$

The first terms of (130) arise from the uniform fields applied at infinity without an inclusion and are given in (125). The remaining terms are due to the presence of elliptic inclusion and are equivalent to (61) with $k = 1$. Thus, all the derivations obtained in previous sections can be employed. Along the elliptic inclusion boundary Γ, (130) is reduced to

$$\begin{aligned}\mathbf{u}|_{\Gamma} &= \mathbf{u}^{\infty}|_{\Gamma}+\cos\psi\mathbf{h}_1-\sin\psi\hat{\mathbf{h}}_1\\ &= a\cos\psi\boldsymbol{\varepsilon}_1^{\infty}+b\sin\psi\boldsymbol{\varepsilon}_2^{\infty}+\cos\psi\mathbf{h}_1-\sin\psi\hat{\mathbf{h}}_1,\end{aligned} \tag{131}$$

$$\begin{aligned}\boldsymbol{\phi}|_{\Gamma} &= \boldsymbol{\phi}^{\infty}|_{\Gamma}+\cos\psi\mathbf{g}_1-\sin\psi\hat{\mathbf{g}}_1\\ &= a\cos\psi\mathbf{t}_2^{\infty}-b\sin\psi\mathbf{t}_1^{\infty}+\cos\psi\mathbf{g}_1-\sin\psi\hat{\mathbf{g}}_1,\end{aligned} \tag{132}$$

in which (49), (65), and (125) are employed.

The solutions inside the piezoelectric inclusion are assumed uniform and have the form

$$\mathbf{u}^{o} = x_1\boldsymbol{\varepsilon}_1^{o}+x_2\boldsymbol{\varepsilon}_2^{o}, \quad \boldsymbol{\phi}^{o} = x_1\mathbf{t}_2^{o}-x_2\mathbf{t}_1^{o}, \tag{133}$$

where

$$\boldsymbol{\varepsilon}_1^{o} = \mathbf{u}_{,1}^{o} = \begin{bmatrix}\varepsilon_{11}^{o}\\ \Omega\\ 2\varepsilon_{13}^{o}\\ -E_1^{o}\end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^{o} = \mathbf{u}_{,2}^{o} = \begin{bmatrix}2\varepsilon_{21}^{o}-\Omega\\ \varepsilon_{22}^{o}\\ 2\varepsilon_{23}^{o}\\ -E_2^{o}\end{bmatrix}, \tag{134}$$

and

$$\mathbf{t}_1^{o} = [\sigma_{11}^{o},\ \sigma_{12}^{o},\ \sigma_{13}^{o},\ D_1^{o}]^{\mathrm{T}} = -\boldsymbol{\phi}_{,2}^{o}, \quad \mathbf{t}_2^{o} = [\sigma_{21}^{o},\ \sigma_{22}^{o},\ \sigma_{23}^{o},\ D_2^{o}]^{\mathrm{T}} = \boldsymbol{\phi}_{,1}^{o}. \tag{135}$$

The constant Ω in $\boldsymbol{\varepsilon}_1^{o}$ and $\boldsymbol{\varepsilon}_2^{o}$ represents the rotation of the x_1-axis in the inclusion. Along the elliptic inclusion boundary Γ, (133) and (49) give

$$\mathbf{u}^{o}|_{\Gamma} = a\cos\psi\boldsymbol{\varepsilon}_1^{o}+b\sin\psi\boldsymbol{\varepsilon}_2^{o}, \quad \boldsymbol{\phi}^{o}|_{\Gamma} = a\cos\psi\mathbf{t}_2^{o}-b\sin\psi\mathbf{t}_1^{o}. \tag{136}$$

The continuity condition states that

$$\mathbf{u}|_{\Gamma} = \mathbf{u}^{o}|_{\Gamma}, \quad \boldsymbol{\phi}|_{\Gamma} = \boldsymbol{\phi}^{o}|_{\Gamma}. \tag{137}$$

Employing (131), (132), and (136) in (137) leads to

$$a\boldsymbol{\varepsilon}_1^\infty + \mathbf{h}_1 = a\boldsymbol{\varepsilon}_1^o, \quad b\boldsymbol{\varepsilon}_2^\infty - \hat{\mathbf{h}}_1 = b\boldsymbol{\varepsilon}_2^o,$$
$$a\mathbf{t}_2^\infty + \mathbf{g}_1 = a\mathbf{t}_2^o, \quad b\mathbf{t}_1^\infty + \hat{\mathbf{g}}_1 = b\mathbf{t}_1^o. \tag{138}$$

In order to solve $\mathbf{h}_1$, $\mathbf{g}_1$, $\boldsymbol{\varepsilon}_1^o, \boldsymbol{\varepsilon}_2^o, \mathbf{t}_1^o$, and $\mathbf{t}_2^o$, two more equations are required. Application of (43) to the matrix and the inclusion yields

$$\mathbf{N}\begin{bmatrix}\boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty\end{bmatrix} = \begin{bmatrix}\boldsymbol{\varepsilon}_2^\infty \\ -\mathbf{t}_1^\infty\end{bmatrix}, \quad \mathbf{N}^o\begin{bmatrix}\boldsymbol{\varepsilon}_1^o \\ \mathbf{t}_2^o\end{bmatrix} = \begin{bmatrix}\boldsymbol{\varepsilon}_2^o \\ -\mathbf{t}_1^o\end{bmatrix}. \tag{139}$$

With (66) we rewrite (138) in matrix form as

$$\begin{bmatrix}\mathbf{h}_1 \\ \mathbf{g}_1\end{bmatrix} = a\begin{bmatrix}\boldsymbol{\varepsilon}_1^o \\ \mathbf{t}_2^o\end{bmatrix} - a\begin{bmatrix}\boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty\end{bmatrix}, \quad \begin{bmatrix}\mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T\end{bmatrix}\begin{bmatrix}\mathbf{h}_1 \\ \mathbf{g}_1\end{bmatrix} = b\begin{bmatrix}\boldsymbol{\varepsilon}_2^\infty \\ -\mathbf{t}_1^\infty\end{bmatrix} - b\begin{bmatrix}\boldsymbol{\varepsilon}_2^o \\ -\mathbf{t}_1^o\end{bmatrix}. \tag{140}$$

Employing (41) and (139), $\boldsymbol{\varepsilon}_1^o, \mathbf{t}_2^o$ are obtained as

$$\left\{\frac{b}{a}\mathbf{N}^o + \begin{bmatrix}\mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T\end{bmatrix}\right\}\begin{bmatrix}\boldsymbol{\varepsilon}_1^o \\ \mathbf{t}_2^o\end{bmatrix} = \left\{\frac{b}{a}\mathbf{N} + \begin{bmatrix}\mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T\end{bmatrix}\right\}\begin{bmatrix}\boldsymbol{\varepsilon}_1^\infty \\ \mathbf{t}_2^\infty\end{bmatrix} \tag{141}$$

assuming that the matrix on the left is non-singular. The constants $\mathbf{h}_1$, $\mathbf{g}_1$ and $\boldsymbol{\varepsilon}_2^o, \mathbf{t}_1^o$ are then computed from $(140)_1$ and $(139)_2$, respectively. The rigid body rotation Ω can easily be determined from $(140)_1$ as

$$\Omega = \frac{(\mathbf{h}_1)_2}{a}. \tag{142}$$

The generalized stress vector along the elliptic inclusion boundary Γ is given by

$$\begin{aligned}\hat{\mathbf{t}}_m = \boldsymbol{\phi}_{,n} &= \frac{\partial\boldsymbol{\phi}|_\Gamma}{\rho(\psi)\,\partial\psi}, \\ &= \frac{-1}{\rho(\psi)}[a\sin\psi\mathbf{t}_2^\infty + b\cos\psi\mathbf{t}_1^\infty] + \frac{-1}{\rho(\psi)}[\mathbf{g}_1\sin\psi + \hat{\mathbf{g}}_1\cos\psi]\end{aligned} \tag{143}$$

when $(132)_2$ is used. Note that the second part of (143) has the same form as $(71)_2$ with $k = 1$. With $(132)_1$ the generalized hoop stress vector is

$$\hat{\mathbf{t}}_n = \boldsymbol{\phi}_{,m} = \boldsymbol{\phi}_{,m}^\infty + [\cos\psi\mathbf{g}_1 - \sin\psi\hat{\mathbf{g}}_1]_{,m}. \tag{144}$$

Observing that $\cos\psi\mathbf{g}_1 - \sin\psi\hat{\mathbf{g}}_1$ is equivalent to $\boldsymbol{\phi}^{\text{II}}|_\Gamma$ given in $(65)_2$ with $k = 1$, its derivative with respect to m can be computed as (Chung, 1995; Hwu and Ting, 1989)

$$\begin{aligned}[\cos\psi\mathbf{g}_1 - \sin\psi\hat{\mathbf{g}}_1]_{,m} = \boldsymbol{\phi}_{,m}^{\text{II}} &= \mathbf{N}_3(\omega)\mathbf{u}_{,n}^{\text{II}} + \mathbf{N}_1^T(\omega)\boldsymbol{\phi}_{,n}^{\text{II}}, \\ &= \mathbf{N}_3(\omega)\frac{\partial\mathbf{u}^{\text{II}}|_\Gamma}{\rho(\psi)\,\partial\psi} + \mathbf{N}_1^T(\omega)\frac{\partial\boldsymbol{\phi}^{\text{II}}|_\Gamma}{\rho(\psi)\,\partial\psi}.\end{aligned} \tag{145}$$

With $\mathbf{u}^{\text{II}}|_\Gamma$ given by $(65)_1$, when $k = 1$, we have

$$\hat{\mathbf{t}}_n = \boldsymbol{\phi}^{\infty}_{,m} - \left\{ \frac{\mathbf{N}_3(\omega)}{\rho(\psi)} [\mathbf{h}_1 \sin\psi + \hat{\mathbf{h}}_1 \cos\psi] + \frac{\mathbf{N}_1^{\mathrm{T}}(\omega)}{\rho(\psi)} [\mathbf{g}_1 \sin\psi + \hat{\mathbf{g}}_1 \cos\psi] \right\}. \tag{146}$$

Again, the second part of (146) is readily obtained by observing that the result stated in $(76)_2$ can be employed with $k = 1$. From (52), (73), and $(125)_2$, $\boldsymbol{\phi}^{\infty}_{,m}$ is found to be

$$\boldsymbol{\phi}^{\infty}_{,m} = \boldsymbol{\phi}^{\infty}_{,1}(-\sin\omega) + \boldsymbol{\phi}^{\infty}_{,2}(\cos\omega) = -\mathbf{t}^{\infty}_2 \frac{b}{\rho(\phi)} \cos\psi + \mathbf{t}^{\infty}_1 \frac{a}{\rho(\psi)} \sin\psi. \tag{147}$$

Combining (146) and (147) yields, with (66),

$$\hat{\mathbf{t}}_n = \frac{-1}{\rho(\psi)} [\mathbf{t}^{\infty}_2 b \cos\psi - \mathbf{t}^{\infty}_1 a \sin\psi] - \frac{1}{\rho(\psi)} \{\mathbf{N}_3(\omega)[\mathbf{h}_1 \sin\psi + (\mathbf{S}\mathbf{h}_1 + \mathbf{H}\mathbf{g}_1)\cos\psi]\}$$
$$- \frac{1}{\rho(\psi)} \{\mathbf{N}_1^{\mathrm{T}}(\omega)[\mathbf{g}_1 \sin\psi + (-\mathbf{L}\mathbf{h}_1 + \mathbf{S}^{\mathrm{T}}\mathbf{g}_1)\cos\psi]\}. \tag{148}$$

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